The Language
for the
Theory of Everything

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Contents

1 Introduction

The study of physics is riddled by a plethora of different fragmented mathematical systems - Gibbs' vectors, matrices, tensors, spinors, differential forms, Hilbert spaces, complex analysis and many, many more. The serious problems this Babel of languages has caused are evident: Many physicists never study Einstein's general relativity due to the difficult tensor algebra it is written in, others do not understand quantum mechanics. Thus, these students do their research without knowing what physicists have dubbed the most important contributions to physics ever. It is thus no surprise there has been no considerable progress in unifying quantum mechanics and general relativity - new research in the one is inaccessible to those in the other. Furthermore, the different mathematical foundations invariably cloud many connections between domains - exactly that which is necessary for potential unification and a Theory of Everything.

I submit that this sorry state of affairs is caused by the physics community's wide adoption of Gibbs' vector algebra (which we still learn at school today) over the turn of the 19th century. Due to its lacking representative power mathematical systems had to be developed on the fly for the new systems that came to be - causing the contemporary fragmentation. This is no accusation to the physicists of that age: The alternative for a vectorial foundation were Hamilton's radically different and unintuitive quaternions.

However, while physicists were vehemently arguing whether to use either Hamilton's quaternions or Gibbs' vectors, the central question should have been, as said by Freeman Dyson: "How can it happen that the properties of three-dimensional space are represented equally well by two quite different and incompatible algebraic structures?"[1]. Hermann Grassmann and William Kingdon Clifford sought to answer this question and developed *Geometric Algebra*, a unification of Hamilton and Gibbs' vectors. It supersedes all mentioned mathematical formalisms - yet staying so simple as to being explainable to anyone.[2] Unfortunately it did not receive the attention it deserved due to Clifford's untimely early death.

The purpose of this essay is to introduce Geometric Algebra and substantiate the many claims of unification made. Moreover, weaknesses of conventional formalisms will be discussed in conjunction with the often surprising insights and simplifications given by Geometric Algebra.

[1] Freeman J. Dyson, Institute for Advanced Study; Missed opportunities, Bulletin of the AMS, v. 78, no. 5, pp. 635 – 652 (1972).

[2] It should be noted, however, that the associated word limit and the goal of this paper makes this a rather concise - and perhaps difficult - read.

2 Inner Product

To introduce Geometric Algebra there are certain prerequisites. One of which, the inner product of two vectors, will be discussed next.

To introduce the inner product, we need simply generalize the scalar product. The scalar product, as we learn at school, has the following properties for vectors a, b, c, scalars r and for arbitrary distinct basis vectors e_i and e_j:

$$\begin{aligned}
a \cdot b &= b \cdot a && \text{(commutativity)} \\
a \cdot (b + c) &= a \cdot b + a \cdot c && \text{(distributivity)} \\
a \cdot (rb) &= r(a \cdot b) && \text{(scalar associativity)} \\
e_i \cdot e_j &= 0 && \text{(definition of orthogonal vectors)} \\
e_i \cdot e_i &= 1 && \text{(preserve Euclidean metric)}
\end{aligned}$$

With these 5 properties the scalar product can be calculated for any two vectors.

For certain parts of physics it is more useful to have a more generalized scalar product called the *inner product*, which is denoted by the same symbol $'\cdot'$ for simplicity. How do they differ? The inner product removes only one of the properties of the scalar product, which is the last one that states that $e_1 \cdot e_1 = 1$.

The reason for this is that ever since Einstein's special theory of relativity, we know that space and time are not absolute, and thus actual space and time is not Euclidean. This last property forces a Euclidean metric onto a space, making it absolute, so it must be removed. An example should illustrate how the inner product can still be used: *Minkowski space*, the space normally used for special relativity, has 4 basis vectors e_1, e_2, e_3, e_4. In this space e_1 is the time coordinate and e_2, e_3, e_4 represent the 3 space coordinates. The rules used instead for the inner product are:

$$e_1 \cdot e_1 = 1$$
$$e_2 \cdot e_2 = e_3 \cdot e_3 = e_4 \cdot e_4 = -1$$

This is often summarized to say that Minkowski space has an inner product with signature $(+, -, -, -)$ or more succinctly $(1, 3)$, which is the notation that will be used in this essay.

3 Outer Product

3.1 Cross Product - and its problems

The other product we learn at school is Gibbs' infamous cross product $\times$. Though looking ingenious at first, it has many weaknesses.

First of all, it is only clearly defined in 3-dimensions. In two-dimensions there is no perpendicular vector; in more than 3-dimensions there are infinitely many other perpendicular vectors. This makes it unsuitable for special relativity which operates in 4 dimensions.

Moreover, the cross product is often used as a work-around for quantities that *really* aren't vectors. Take for example angular momentum L, a quantity that measures angular acceleration about an axis which is described by the equation:

$$L = r \times p \tag{1}$$

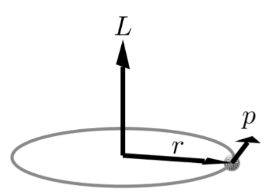

Figure 1: The angular momentum vector L for a particle rotating around a point (all figures have been created by the author using Manim unless specified otherwise)

It is clear from the image that this vector L is an unfortunate definition for angular momentum. The vector does not point in the direction in which we are rotating - and its magnitude rather unintuitively depends on the speed and mass of the object rotating. In two dimensions we'd also be introducing an unnecessary dimension pointing out of the page - for a strictly 2-dimensional phenomenon! Moreover, one must use the right-hand rule to figure out geometrically in what direction the vector is pointing, and any student can attest that the right-hand rule is not only difficult to remember, but also simply confusing - especially since the left-hand rule is occasionally necessary as well.

But matters get worse than the merely geometrically unaesthetic properties just introduced. L is not actually a vector, it is a *pseudovector*. Normally, it acts like a vector but when rotated or mirrored it acts differently and sometimes has a sign change! There are many other physical quantities with such problems like torque τ, the magnetic field vector $\vec{B}$ and more.

The central problem here is representing angular momentum by a vector. Really, it is a kind of oriented area, so we would want some object L looking something like an area (see figure 3). Gibbs' vectors do not allow one to encode areas directly, generally one uses a vector perpendicular to it to represent it. Evidently this is an ugly workaround, so why not just encode areas directly? For this, I must introduce you to the outer product.

3.2 Introducing the solution

The outer product - first discovered by Hermann Grassmann in his renowned *Ausdehnungslehre* published in 1844, will aid us in this endeavour. The outer product $a \wedge b$ of two vectors encodes the oriented area between them, creating a so-called *bivector*:

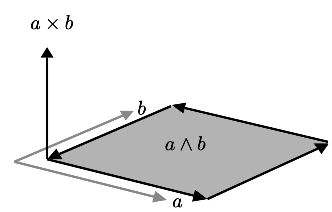

Figure 2: The outer product $a \wedge b$ and the cross product $a \times b$

What's the use of it being an *oriented* area? Well, take for example our angular momentum, where it would encode the direction of the rotation:

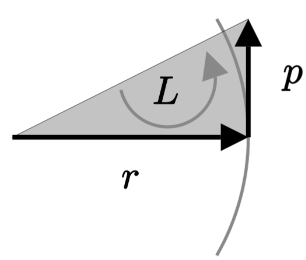

Figure 3: The oriented area or bivector L

Due to the orientation we can conclude for any two vectors that the following equation must be true (we say that the outer product is *anticommutative*)

$$a \wedge b = -b \wedge a \tag{2}$$

If a is parallel to b, then we have:

$$a \wedge b = 0 \tag{3}$$

This also makes sense geometrically:

Figure 4: If a and b are parallel, they do not span an area.

Bivectors can be added to each other in a manner reminiscent of normal vectors: This means that the outer product is *distributive*:

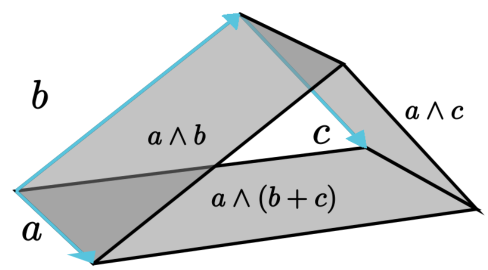

Figure 5: Addition of bivectors

$$a \wedge (b + c) = a \wedge b + a \wedge c \tag{4}$$

One further property of the outer product is that it is *associative*, which is made evident by the fact that these describe the same 3-dimensional oriented volume (called a *trivector*):

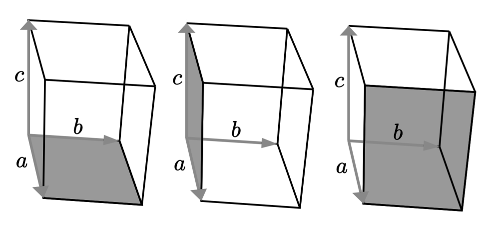

Figure 6: Three trivectors all representing $a \wedge b \wedge c$

Thus

$$a \wedge (b \wedge c) = (a \wedge b) \wedge c \tag{5}$$

We now have 0-dimensional objects we call the real numbers, 1-dimensional objects we call vectors and 2-dimensional objects called bivectors. This pattern can be continued. $a \wedge b \wedge c$ is a trivector or an oriented volume. More generally, an n-vector is the outer product of n vectors, and thus represents an n-dimensional analogue of a *hypervolume*. One thing I want to note at this point is that bivectors, trivectors, etc. do not have a "shape", so they are not necessarily parallelograms but could have any form: a circle, amoeba, tesseract - it does not really matter. The representation using parallelograms is simply convenient because they have the same area as the associated parallelogram.

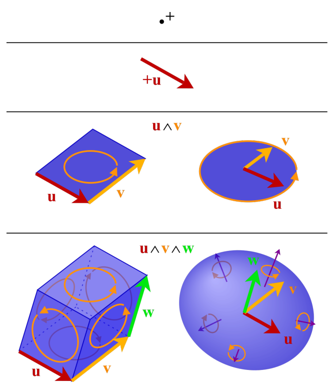

Figure 7: The different geometric objects contained in Geometric Algebra. Maschen, (2014, April 1st) *N-Vector*, retrieved from `https://en.wikipedia.org/wiki/Exterior_algebra#/media/File:N_vector_positive.svg`

An arbitrary element of Geometric Algebra is called a *multivector*. For a 3-dimensional Geometric Algebra a multivector can thus be decomposed as follows:

$$M = \alpha + v + B + T \tag{6}$$

where α is a real number, v a vector, B a bivector and T a trivector (due to the abundance of vectors in this essay, vectors will be denoted without the arrow). We say that real numbers are of grade 0, vectors grade 1, bivectors grade 2 and more generally that k-vectors are of grade k.

Now you may rightly think: *Hold on a second, are you adding a vector to a bivector right there? And a number there as well? That's not allowed!* However, you can have unlike things be together. We have this for example with complex numbers $z = a + bi$ which have a complex and imaginary part. Another useful analogy is thinking that you cannot add an apple to a pear to give you 2 pears, but you can still have an apple alongside a pear. They are simply two different parts of your lunch snack.

4 Geometric Product

The careful reader may have noticed that the inner product and outer product are counterparts. The inner product lowers the grade of two objects, the outer product raises them. The inner product gives information about the parallelity of the two vectors involved, while the outer product gives information about their orthogonality. William Kingdon Clifford noticed this and tried to use both of their features. Thus he defined the fundamental product of Geometric Algebra, the geometric product.

For arbitrary multivectors $A, B \in \mathcal{G}^n$, where $\mathcal{G}^n$ is an n-dimensional Geometric Algebra, the geometric product is defined with the following properties:

1. $A(BC) = (AB)C$ (associativity)
2. $A(B + C) = AB + AC$ and $(B + C)A = BA + CA$ (distributivity)
3. $ab = a \cdot b + a \wedge b$, where a, b are vectors (Fundamental Identity)

The first two properties are quite normal, but the last requires some thinking. It is called the *Fundamental Identity*, and is the central idea from Geometric Algebra. Some may be bugged by the addition of a scalar $a \cdot b$ and a bivector $a \wedge b$. We want to note again our discussion in the previous chapter - there's nothing wrong with this, we can add scalars to bivectors. Another thing to be noted about the geometric product is that it is not necessarily commutative, i.e. $ab \neq ba$. There's nothing wrong with that either, indeed the cross product itself is not commutative. There are many processes that are not commutative - I do not recommend putting on socks after shoes, for instance.

Indeed, we will see this product is extremely powerful in this essay, but first we prove some theorems for convenience:

Theorem 1 (The inverse of a vector)**.** *The inverse of a vector a with respect to the geometric product is:*

$$a^{-1} = \frac{a}{aa} = \frac{a}{a \cdot a} \tag{7}$$

This may seem obvious, but not every product necessarily has an inverse. Examples of this are the scalar and cross product.

Proof. First, due to the fundamental identity in step 1:

$$a^2 = aa \stackrel{1}{=} a \cdot a + a \wedge a \stackrel{2}{=} a \cdot a \tag{8}$$

In step 2 we use the fact that $a \wedge a = 0$ because it swoops out no area. Because $a^2 = a \cdot a$, this implies that a^2 is a scalar. Now we can prove the theorem. Using our definition for a^{-1}:

$$aa^{-1} = a\frac{a}{a \cdot a} = \frac{a \cdot a}{a \cdot a} = 1 \tag{9}$$

□

This is quite profound: Neither the inner product nor the outer product have an inverse, but combined they do! Indeed we will see this operation is very useful.

We now prove a further theorem to help us in manipulating expressions later on and that will clarify the term *geometric* in *geometric product*.

Theorem 2. *If vectors a,b are parallel, then*

$$ab = ba, \textit{i.e. the vectors commute} \tag{10}$$

Proof. Recall that $ab = a \cdot b + a \wedge b$. Because a, b are parallel, we know that $a \wedge b = 0$ as they sweep out no area. Thus:

$$ab = a \cdot b + a \wedge b = a \cdot b = b \cdot a = b \cdot a + b \wedge a = ba \tag{11}$$

□

Theorem 3. *If vectors a,b are orthogonal, then*

$$ab = -ba, \textit{i.e. the vectors anticommute} \tag{12}$$

Proof. For orthogonal vectors we know that $a \cdot b = 0$. Thus:

$$ab = a \cdot b + a \wedge b = a \wedge b = -b \wedge a = -b \wedge a - b \cdot a = -ba \tag{13}$$

□

We now introduce a useful corollary that will help a lot in manipulating equations later.

Corollary 1. *If e_i and e_j are two different orthogonal basis vectors, then*

$$e_i e_j = -e_j e_i \tag{14}$$

Proof. This follows directly out of the last theorem, as e_i and e_j are orthogonal. □

This last equation is probably the most important and will be referred to more often throughout this essay.

With the mathematical foundation of Geometric Algebra (abbreviated GA) down, we can now turn to rotors - one of the most useful developments of the algebra.

5 Rotors

Rotations are extremely important in physics. The quaternions mentioned in the introduction were great at handling them - Gibbs' vectors (that we learn at school) arguably are not. However, quaternions are hindered by the fact that they are 4-dimensional - while 3-dimensional rotations definitely do not need 4 dimensions. Moreover, they seem like a kind of 'black-box' - not yielding easily to a geometric interpretation and not being usable in higher dimensions. GA features a much better device for rotating, the so-called *rotor*. First, however, we will have to deal with reflections.

5.1 Reflections

Let us try to reflect a vector v across vector m to v_{ref}. The formula for this is as follows:

$$v_{\text{ref}} = mvm^{-1} \tag{15}$$

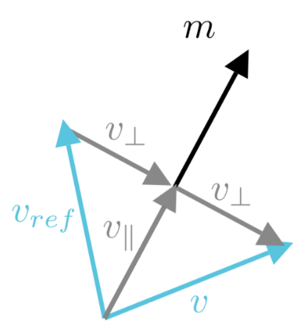

Figure 8: v is reflected across m to v_{ref}

Proof. Let $v_\parallel$ be the component of v parallel to m, and $v_\perp$ that which is perpendicular, so $v = v_\parallel + v_\perp$. Then:

$$\begin{aligned} mvm^{-1} &= m(v_\parallel + v_\perp)m^{-1} \\ &= mv_\parallel m^{-1} + mv_\perp m^{-1} \\ &\stackrel{1}{=} v_\parallel mm^{-1} - v_\perp mm^{-1} \\ &= v_\parallel - v_\perp \end{aligned}$$

We used the fact that parallel vectors commute and orthogonal vectors anticommute in (1). The last expression is exactly the equation for a normal reflection, though note that ours wins in strength - it does not require a decomposition. This is a general feature of GA - one rarely needs to look at the components of a vector or other object unless in a specific use case. □

5.2 Rotations

We now extend reflections to rotations. First we choose two *unit* vectors m and n to reflect vector v by. By using our theorem the reflection across n and then across m is thus:

$$mnvn^{-1}m^{-1} = (mn)v(mn)^{-1} \tag{16}$$

If we look at figure 8, we see that this is actually a rotation in the plane $m \wedge n$ with angle 2θ (where θ is the angle between m and n). Thus mn generates a

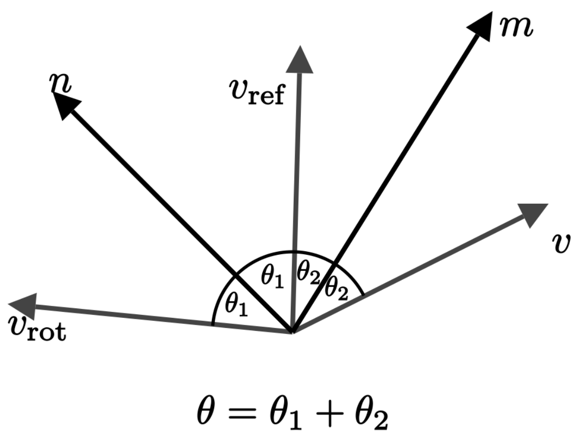

Figure 9: v is reflected across m to v_{ref}, and then across n to v_{rot}

rotation if we do a "sandwich product". Let's change mn a bit. Let B be the unit bivector representing the plane $m \wedge n$. Recall that m and n are unit vectors:

$$mn = m \cdot n + m \wedge n = |m||n| \cos\theta + B \sin\theta = \cos\theta + B \sin\theta \tag{17}$$

On the right we have a more useful form because choosing the two vectors along which we reflect to achieve a certain rotation is quite complicated. However, we're not done yet. mn rotates v by 2θ and into the wrong direction (by convention rotations are unfortunately performed counter-clockwise). To fix that, we thus define a *rotor* R for a rotation in the plane B by the angle θ thusly:

$$R = \cos\left(\frac{\theta}{2}\right) - B \sin\left(\frac{\theta}{2}\right) \tag{18}$$

The equation to rotate a vector, or actually any multivector M, is thus:

Theorem 4. *Rotor equation*

$$M_{rot} = RMR^{-1} \tag{19}$$

We note one thing: The fact we are rotating in a plane effectively lets us generalize the concept of rotation, which is normally done around an axis of rotation. However, the axis of rotation is a 3-dimensional concept only - in two dimensions the axis of rotation would be in the non-existent third dimension, while in dimensions higher than 3 (like in special relativity) an axis of rotation doesn't exist.

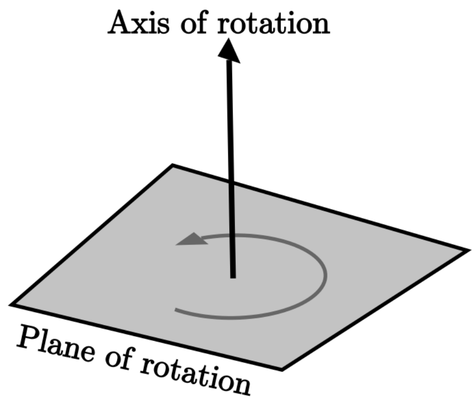

Figure 10: The difference between the axis of rotation and the more generalizable plane of rotation

5.3 Spinors

Spinors have a long history of being utterly confusing - and are also coincidentally where GA has been rediscovered (albeit in different forms) by physicists the most. Spinors are characterized by the seemingly strange property that a full rotation of 360° turns a spinor negative, so it is suddenly different! In the following, we will show a definition of a spinor using rotors in GA that illuminates some of spinors' odd properties.

A spinor (most oftenly denoted by ψ) is of the following form:

$$\psi := aR \tag{20}$$

where $a \in \mathbb{R}$ and R is a rotor.

Thus, spinors are simply rotors multiplied by a constant. At first, this may seem rather odd, what's the use of that? But we will see later that it is extremely useful.

The first property we can discuss here is the weird way spinors act under rotations. First, we note we can decompose any vector v as follows:

$$\begin{aligned} v &= \psi e_3 \psi^{-1} && (21)\\ &= aRe_3aR^{-1} && (22)\\ &= a^2 \underbrace{Re_3R^{-1}}_{\text{Rotation}} && (23) \end{aligned}$$

Here we used the decomposition of the spinor ψ from (20). How is this a vector? Well, it should be noted that we are rotating from an arbitrary basis vector (here e_3) using the rotor equation from the previous chapter, and then dilating this vector - so we indeed do get a vector, in fact any vector we can think of.

Next, imagine we rotate this vector v using another rotor Q to get the rotated vector v_{rot}. We now have:

$$v_{\text{rot}} = QvQ^{-1} \tag{24}$$

Since every vector has a decomposition like in (20), how does the spinor ψ transform so that it works for v_{rot}?

$$\begin{aligned} v_{\text{rot}} &= QvQ^{-1} && (25)\\ &= Q\psi e_3\psi^{-1}Q^{-1} && (26)\\ &= (Q\psi)e_3(Q\psi)^{-1} && (27) \end{aligned}$$

If we now decompose v_{rot} into spinors as in (21):

$$\begin{aligned} v_{\text{rot}} &= \psi_{\text{rot}} e_3 \psi_{\text{rot}}^{-1} && (28)\\ &= Q\psi e_3(Q\psi)^{-1} && (29) \end{aligned}$$

Comparing the top two equations we now get the transformation law for spinors under rotations:

$$\psi_{\text{rot}} = Q\psi \tag{30}$$

This is rather odd as spinors 'rotate' by being multiplied on the left side by a rotor, but not by its reverse on the right like in the rotor equation! However, Geometric Algebra shows this is quite normal. Spinors aren't truly geometric primitives, so speaking of them as 'rotating' is quite misleading. Really, they are unnormalized rotors, objects that act on more primitive objects, so it is no surprise they act differently when we rotate their original vectors!

At the beginning of the chapter we mentioned that a full rotation by 360° gives some interesting results for spinors. We will now explore that. Using equation (30) and setting the rotor such that $\theta = 360°$, we get:

$$Q = \cos\frac{\theta}{2} - B\sin\frac{\theta}{2} \tag{31}$$

$$= \cos\frac{360°}{2} - B\sin\frac{360°}{2} \tag{32}$$

$$= -1 - B\sin(180°) = -1 \tag{33}$$

$$\therefore \psi_{\mathrm{rot}} = Q\psi = -\psi \tag{34}$$

And so we see that spinors become negative under a 360° rotation!
For anyone who has studied spinors in the conventional way, the ease with which Geometric Algebra deals with and explains spinors is surprising. It is rich with geometric interpretation and, frankly, makes much more sense. In fact, the Geometric Algebra description of spinors yields even more than the traditional way they are introduced, as we will see as we start using this tool.

5.4 Complex numbers

The importance of complex numbers to physics cannot be understated, yet the imaginary unit i is still accepted at face value for the most part. In the next chapter, we will deal with this popular number.

The astute observer may have noticed that the definition of a rotor $R = \cos\frac{\theta}{2} - B\sin\frac{\theta}{2}$ has similarities to Euler's identity $e^{i\theta} = \cos\theta + i\sin\theta$. Complex numbers, like rotors, are also used to rotate objects very often. However, complex numbers - though one quickly gets used to them - are still rather mysterious. Perhaps it comes as a surprise then that complex numbers are simply spinors in two-dimensional geometric algebra! Let us substantiate this claim.

In $\mathcal{G}(2,0)$ (the 2-dimensional geometric algebra with signature $(2,0)$), there is only one plane of rotation represented by e_1e_2. Thus we know for the unit bivector that $B = e_1e_2$ and we get:

$$\psi = aR = a\cos\frac{\theta}{2} - ae_1e_2\sin\frac{\theta}{2} \tag{35}$$

This looks quite similar to the polar form for complex numbers:

$$c = re^{i\theta} = r\cos\theta + ri\sin\theta \tag{36}$$

Indeed we will show they are equivalent. To do that we first note that for what we want to do we need only show that the bivector B = e_1e_2 squares to -1 like the imaginary unit i. The fact that we have $\frac{\theta}{2}$ and a negative sign do not matter as changing θ and the constant a can fix this.

$$\begin{aligned}
B^2 &= (e_1e_2)^2 && (37)\\
&= e_1e_2e_1e_2 && (38)\\
&= e_1(e_2e_1)e_2 \qquad\qquad \mid e_2e_1 = -e_1e_2 && (39)\\
&= -e_1(e_1e_2)e_2 && (40)\\
&= -(e_1)^2(e_2)^2 \qquad\qquad \mid e_i^2 = e_i \cdot e_i = 1 && (41)\\
&= -1 && (42)
\end{aligned}$$

This gives us $B^2 = -1$. Thus, the unintuitive geometry and behaviour of the imaginary unit i can be explained by the fact that it is simply a bivector! There is nothing imaginary about it, it is a very real oriented area. No wonder that Gauss' initial venture to use the complex $a + bi$ as a vector did not work:[3] It is really a scalar plus bivector, a spinor.

Imaginary numbers have many uses in physics - the most problematic of which is in quantum mechanics. Unfortunately the way they are used they do not correspond to real objects - which makes quantum mechanics all the more confusing and abstract. Indeed, the mathematical physicist David Hestenes has said that imaginary numbers in quantum mechanics no longer need be used - they are incorporated in geometric algebra and given ample geometric intuition - and with it new directions for research.

[3]Orlando Merino, *A Short History of Complex Numbers*, University of Rhode Island, January, (2006)

5.5 Quaternions

We now face Hamilton's famous numbers, the contestant to Gibbs' vector algebra - the quaternions. Given that the rotor is good at handling rotations, we would expect quaternions also to be incorporated in geometric algebra.

Let us look at spinors in $\mathcal{G}(3,0)$. First of all, we note that bivectors in 3 dimensions are spanned by the unit bivectors e_1e_2, e_2e_3 and e_1e_3. To more easily work with them, we define i, j, k:

$$i = e_1e_2, j = e_2e_3, k = e_1e_3 \tag{43}$$

Thus, a spinor ψ is of the form:

$$\begin{aligned} \psi &= aR & (44)\\ &= a\cos\frac{\theta}{2} - B\sin\frac{\theta}{2} & (45)\\ &= a\cos\frac{\theta}{2} - (bi + cj + dk)\sin\frac{\theta}{2} & (46) \end{aligned}$$

Let's simplify how this looks right now. We can always choose α, β, γ and ϵ such that

$$\psi = \alpha + \beta i + \gamma j + \epsilon k \tag{47}$$

We also note that as discussed in the previous chapter, we have

$$\begin{aligned} i^2 &= (e_1e_2)^2 = -1\\ j^2 &= (e_2e_3)^2 = -1\\ k^2 &= (e_1e_3)^2 = -1 \end{aligned}$$

We now compute the following few identities:

$$\begin{aligned} ij &= (e_1e_2)(e_2e_3) = e_1e_2e_2e_3 &&= e_1(e_2^2)e_3 = e_1e_3 = k\\ jk &= (e_2e_3)(e_1e_3) = -e_2(e_3^2)e_1 &&= -e_2e_1 \quad = e_1e_2 = i\\ ki &= (e_1e_3)(e_1e_2) = -(e_1)^2e_3e_2 &&= -e_3e_2 \quad = e_2e_3 = j \end{aligned}$$

These are exactly the defining equations for quaternions! Thus we see that quaternions are simply three-dimensional spinors, no wonder they perform rotations well! We also see that quaternions very much are three-dimensional objects - and really shouldn't be thought of as 4-dimensional as Hamilton believed and still is taught today. Furthermore, this explains why Hamilton's goal of using the non-scalar $(bi + cj + dk)$ part of quaternions as vectors did not work - it is actually a bivector. Indeed, it seems we have answered Freeman Dyson's question stated in the introduction: GA explains why both quaternions and Gibbs' vectors can describe ordinary geometry, regardless of their quite different foundations.

6 Quantum Mechanics

Quantum mechanics is the study of the extremely small. Perhaps even more important than Einstein's Relativity, it is one of the most ground-breaking theories ever to have trodden the world. It is famously unintuitive - Richard Feynman even saying "I think I can safely say that nobody understands quantum mechanics". In contrast to all prior deterministic theories it is also probabilistic - we cannot know for certain what the outcome of an experiment is. This sounds like a limitation, yet quantum mechanics is the theory that most accurately reflects and predicts experimental results. Indeed, the mathematical results of quantum mechanics are very clear. However, the interpretation of these equations is still highly debated - there are more than 15 different philosophical interpretations of them. What Geometric Algebra can address in this debate we will explore next.

One of the simplest cases where quantum effects differ from classical physics is in the theory of spin-1/2 particles. Spin-1/2 particles are extremely fundamental and important in physics - it is the effects from spin that underlie the categorization of all particles into either fermions or bosons, the periodic table in chemistry, or to a certain extent the existence of mass. Moreover, spin proved necessary for unifying quantum mechanics and special relativity in the Dirac equation. It is thus possibly surprising to know that spin is not very well understood.[4]

Spin-1/2 is a property of many particles, electrons, protons, neutrons, quarks and neutrinos, and rather confusing, since these particles cannot *actually* be spinning, as they are thought of as points. Moreover, they would be spinning at many times the speed of light - quite an illegal activity according to Einstein. Another defining (and albeit confusing) feature of 'spin-1/2' is its spinor that becomes negative under a 360° rotation. Recalling the discussion from the previous chapter we thus expect Geometric algebra to illuminate this mostly highly confusing area of physics.

With the necessary background knowledge out of the way, we will first show the tools quantum mechanics (abbreviated QM) conventionally uses. Most important in this regard are the *Pauli matrices* σ_1, σ_2 and σ_3:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{48}$$

They are rather mysterious - especially considering the imaginary unit in σ_2. In fact, this is one of the reasons many objects in QM are considered not real but abstract - because we cannot assign a truly geometric meaning to the imaginary unit i, nor easily to matrices either.

[4]The mathematical physicist Michael Atiyah has even noted that: "No one understands spinors. Their algebra is formally understood but their general significance is mysterious." cited from Farmelo, Graham *The Strangest Man: The hidden life of Paul Dirac, quantum genius.* (2009) Faber Faber. p. 430

The Pauli matrices also have some more properties:

$$\sigma_1\sigma_2 = -\sigma_2\sigma_1$$
$$\sigma_2\sigma_3 = -\sigma_3\sigma_2$$
$$\sigma_1\sigma_3 = -\sigma_3\sigma_1$$

As well as

$$\sigma_1^2 = \sigma_2^2 = \sigma_3^2 = I$$

where I is the identity matrix. The attentive reader may have noticed these relations are quite similar to the relations governing the behaviour of the basis vectors e_1, e_2, e_3 in $\mathcal{G}(3,0)$. Indeed, much of QM can instead be done with e_1, e_2 and e_3 as they of course fulfill the following equations:

$$e_1e_2 = -e_2e_1$$
$$e_2e_3 = -e_3e_2$$
$$e_1e_3 = -e_3e_1$$

$$e_1^2 = e_2^2 = e_3^2 = 1$$

The crucial distinction is that the Pauli matrices are operators in quantum isospace, whereas the e_k are vectors in real space. The latter are much more geometrically understandable. Moreover, multiplication of Pauli matrices does not clearly give us an area, while we know that $e_1e_2 = e_1 \wedge e_2$ has a clearly geometrical interpretation.

There is another more philosophical problem Geometric Algebra addresses. In general, the Pauli matrices are thought to be intrinsically related to spin. However, they are analogous to basis vectors which can be used for much more than just spin! This brings quantum mechanics much closer to classical mechanics as well as to relativity - as these have already been formulated in $\mathcal{G}(3,0)$ (or for special relativity in $\mathcal{G}(1,3)$) by David Hestenes in his *New Foundations for Classical Mechanics* and *Spacetime Algebra*.

The insights do not end there, however. The discussion of spin for electrons is normally done with a two-component *spinor* with complex numbers α and β:

$$|\psi\rangle = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \tag{49}$$

This term isn't very telling, however. Indeed it has the odd property that rotation by $360°$ turns it negative. The discussion of spinors in the previous chapter should alert us that GA may be able to clear things up here. Indeed, using the following identification between traditional QM and its treatment in GA (where $\leftrightarrow$ denotes the equivalent expression in GA instead of complex matrix algebra)

$$|\psi\rangle = \begin{pmatrix} a_0 + ia_1 \\ a_2 + ia_3 \end{pmatrix} \leftrightarrow \psi = a_0 + a_1e_1e_2 + a_2e_1e_3 + a_3e_2e_3 \tag{50}$$

It is easy to show that operations on the traditional wave function can be translated to GA (and we shall do this a bit later). We see that the Geometric Algebra ψ is indeed simply a 3-dimensional spinor as we have discussed extensively in chapter 5! Thus it is also no surprise that it rotates in the special way as we discussed in the previous chapter. However, in its matrix form this is perfectly unclear. Indeed spinors are often regarded as this intrinsically quantum mechanical phenomenon, but this is simply untrue. Our discussion in chapter 5 never once required quantum mechanical phenomena.

Now that we know that ψ is a GA spinor, we can define the *spin vector* s as we did in chapter 5, though multiplying it by the scalar $\frac{1}{2}\hbar$ as spin is generally measured in this unit:

$$s = \frac{1}{2}\hbar\psi e_3\psi^\dagger \tag{51}$$

We use e_3 instead of e_1 because the e_3 or $z-$axis is often used as a reference point in QM. With this we have however not completely shown that QM can be translated. An important operation in QM is the multiplication of a traditional spinor $|\psi\rangle$ by a 2×2 matrix. These matrices allow one to calculate the particle's momentum, position, energy and more. We will now convert this matrix multiplication - completing the theoretical translation into GA.

Every 2×2 matrix A can be decomposed with the Pauli matrices and the identity matrix I:

$$A = a_0 I + a_1\sigma_1 + a_2\sigma_2 + a_3\sigma_3 \tag{52}$$

(where $a_i \in \mathcal{C}$)

The reason this works is because a complex 2×2 matrix has $2\times 2\times 2 = 8$ degrees of freedom, and the 4 complex numbers on the right hand side of the equation also have 4×2 degrees of freedom.

So, if we can translate multiplication by the identity I and the Pauli matrices, we are done, as all other matrix multiplications can then be composed from that.

First we note that multiplication by the identity leaves the vector the same. Thus translation into GA is quite trivial:

$$I\,|\psi\rangle = |\psi\rangle \leftrightarrow 1\psi = \psi \tag{53}$$

Now for the Pauli matrices. First, we note for σ_1:

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{pmatrix} a_0 + ia_1 \\ a_2 + ia_3 \end{pmatrix} = \begin{pmatrix} a_2 + ia_3 \\ a_0 + ia_1 \end{pmatrix} \leftrightarrow a_2 + a_3 e_{12} + a_0 e_{13} + a_1 e_{23} = e_1\psi e_3 \tag{54}$$

Here we used the translation from (50). It should be noted that the use of e_3 is arbitrary, any unit vector could be used. However, the $z-$axis is often chosen as a frame of reference in QM, so the same is done in GA. It should be noted this does not sacrifice coordinate independence. Similar relations are true for σ_2 and σ_3, however these calculations are completely analogous and trivial. Finally we get the following identifications:

$$\sigma_1 |\psi\rangle \leftrightarrow e_1 \psi e_3$$
$$\sigma_2 |\psi\rangle \leftrightarrow e_2 \psi e_3$$
$$\sigma_3 |\psi\rangle \leftrightarrow e_3 \psi e_3$$

This was a rather formal exercise and not of much theoretical significance for us just yet, except that it shows that all of QM can be done in GA. However, David Hestenes and his students have reformulated wide swaths of QM from the 1970s onwards and the geometrical insights given by a reformulation into GA are profound. Due to these insights David Hestenes has since reproposed Schrödinger's forgotten *Zitterbewegung* (though with a nontrivial twist) as an interpretation of electron spin.[5] We thus see that translation into such a powerful geometric language is not for naught - it gives valuable avenues for future research and interpretation of known results. Moreover, if GA is used in classical mechanics (as it can be very nicely done, though that is not the purpose of this essay), one need not introduce further mathematical systems - thereby greatly making the physics curriculum more efficient.

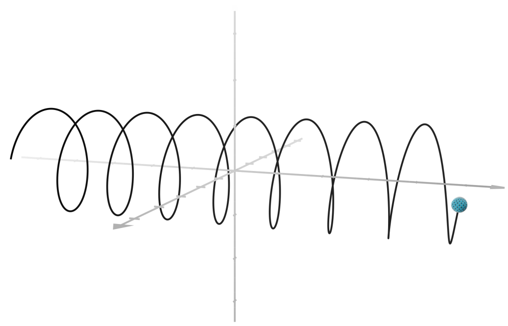

Figure 11: The proposed Zitterbewegung interpretation of electron spin

[5] *The Zitterbewegung Interpretation of Quantum Mechanics*, David Hestenes (Found. Physics., Vol. 20, No. 10, (1990) 1213–1232.)

7 Dirac matrices and Special Relativity

One of the most important equations in the description of quantum phenomena is the Dirac equation. Being the first equation to postulate antimatter and unify special relativity and quantum mechanics, the insights that followed it are put fully on par with the works of Newton, Maxwell, and Einstein before him. Paul Dirac, its inventor, initially only sought to extend the formerly discussed Pauli theory to also work with Einstein's special relativity. To do this, he used the Dirac matrices:

$$\begin{aligned} \gamma_0 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad \gamma_1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \\ \gamma_2 &= \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \quad \gamma_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{55}$$

Evidently, they are rather unwieldy, nor are they easily physically interpretable. The γ_0 matrix is different than the others, and is often associated with the time dimension, the other three with the 3 dimensions of space. The 4 matrices have some peculiar properties, however, that explain why they are even used in the first place:

$$\begin{gathered} \gamma_0^2 = I \\ \gamma_1^2 = \gamma_2^2 = \gamma_3^2 = -I \\ \gamma_i\gamma_j = -\gamma_j\gamma_i \end{gathered}$$

Once again we note some similarities with geometric algebra. Indeed, the geometric algebra $\mathcal{G}(1,3)$ has exactly these properties - so the algebra created by the Dirac matrices is isomorphic to it! Using the basis vectors e_0, e_1, e_2, e_3:

$$\begin{gathered} e_0^2 = 1 \\ e_1^2 = e_2^2 = e_3^2 = -1 \\ e_ie_j = -e_je_i \end{gathered}$$

Thus, we can now assign a geometric picture to the Dirac matrices, which are actually just matrix representations of the geometric algebra $\mathcal{G}(1,3)$. No physicists, including Dirac, thought they were actually vectors, showing just how strongly matrices can obfuscate geometric meaning. Indeed, Dirac matrices are popularly believed to be inherently quantum mechanical, though we see, just as with the Pauli matrices, that it is not necessarily so. $\mathcal{G}(1,3)$, the geometric algebra that is created with these 4 basis vectors, is better known as *Spacetime algebra*, and may remind you of the Minkowski metric we discussed in the chapter introducing the inner product.

Spacetime algebra turns out to be an extremely useful algebra for both quantum mechanics and traditional special (and general) relativity. In this essay we can only hint towards the intriguing developments (as, unfortunately, we cannot deal with all of physics in an essay). Nonetheless, it has for example been used to unify the 4 Maxwell equations into one (in the following succinct form $\nabla F = J$) It should be noted that simpler Maxwell equations were the original reason why the Gibbs' vectors were used over the quaternions - Maxwell originally had more than 20 quaternionic equations, and Heaviside brought this down to 4 using Gibbs' vectors. Now it has been brought down to one. To achieve this unification, a coordinate-independent *Geometric Calculus* has been developed that unifies many theorems of complex analysis under one *Fundamental theorem of calculus*. Once again a trend of simplification and unification can be observed.

Furthermore, in spacetime algebra a real Dirac equation without imaginary numbers has been formulated, with new features formerly never identified as research opportunities. Spacetime algebra has been used as the basis for a synthetic treatment of general relativity - in this line of research Doran and Lasenby's *Gauge Theory of Gravity* - simpler than its predecessors and written in Geometric Algebra - should be noted. Since spacetime algebra and more generally the field of Geometric Algebra have only received attention in the past few decades only, it is reasonable to assume many of the intriguing developments are still waiting to be made.

8 Conclusion

The shear breadth of topics discussed in this essay demonstrates the widespread unification of fragmented mathematical physics achieved by GA, which will greatly enhance communication and transfer of knowledge across disciplines. We have shown that Geometric Algebra supersedes Gibbs' vectors and Hamilton's quaternions. Geometric Algebra has been successfully applied in both special relativity and quantum mechanics, finally unifying the mathematics and providing a mathematical bridge between the two theories. This translation was not only one of renaming symbols - we gained many new insights using GA concerning the role of Gibbs' vectors, pseudovectors, quaternions, spinors but also the mysterious nature of the Pauli and Dirac matrices as well as the imaginary unit i. Along the way we discovered many new avenues for research, showing that understanding the geometry behind the physics is critical for the understanding of said physical phenomena, which will aid in the quest for unification.

With such an abundance of unification and insight, perhaps, we truly can announce we have found the language for the Theory of Everything.

9 Bibliography

Dyson, Freeman *Missed opportunities.* Bulletin of the AMS, v. 78, no. 5, pp. 635 – 652. 1972.

Doran, Chris and Anthony Lasenby. *Geometric Algebra for physicists.* Cambridge, UK: Cambridge University Press, 2003

Graham, Farmelo *The Strangest Man: The hidden life of Paul Dirac, Quantum Genius.* Faber Faber. 2009.

Hestenes, David. *Spacetime Algebra.* New York, USA: Gordon and Breach Science Publishers, 1966

Hestenes, David. *Clifford Algebra to Geometric Calculus: A Unified Language for Mathematics and Physics.* Cham, Switzerland: Springer, 1984 edition

Hestenes, David. *The Zitterbewegung Interpretation of Quantum Mechanics.* Found. Physics., Vol. 20, No. 10, 1213–1232. 1990.

Josipovic, Miroslav. *Geometric Multiplication of Vectors.* Cham, Switzerland: Birkhäuser, 2019

MacDonald, Alan. *Linear and Geometric Algebra.* Decorah, USA: Luther College, 2010 Edition.

Merino, Orlando. *A Short History of Complex Numbers.* University of Rhode Island, USA: 2005